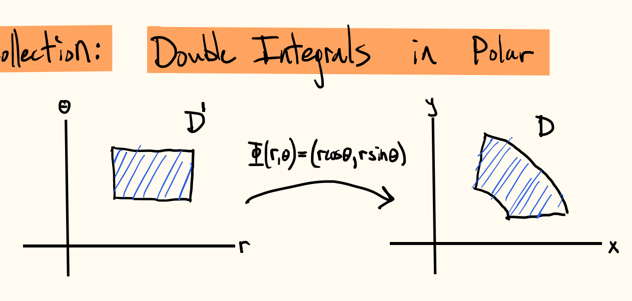
Meeting 13 Math 22

Recollection: Double Integrals in Polar



Suppose D' is transformed into D by the polar change of variables $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\iint_D f(x,y)\,dA = \iint_{D'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

Activity 11.5.4

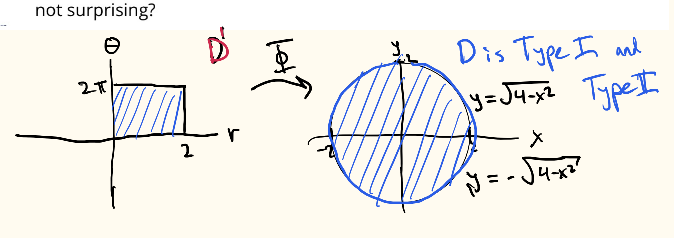
- Complete together as a class.

Activity 11.5.4. Let $f(x,y) = x + y$ and $D = \{(x,y) : x^2 + y^2 \le 4\}$.

a. Sketch the region D and then write the double integral of f over D as an iterated integral in rectangular coordinates.

b. Write the double integral of f over D as an iterated integral in polar coordinates.

c. Evaluate one of the iterated integrals. Why is the final value you found not surprising?



(a) $\iint_D x+y\,dA \overset{\text{Type I}}{=} \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x+y\,dy\,dx$

The transformation turns the rectangle

$$D' = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

into the circle $D = \{(x,y) \mid x^2+y^2 = 4\}$

(b)
$$\iint_D x+y\,dA = \int_0^{2\pi}\int_0^2 (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 r^2(\cos\theta + \sin\theta)\,dr\,d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{1}{3}r^3(\cos\theta + \sin\theta)\right]_0^2 d\theta$$
$$= \frac{8}{3}\int_0^{2\pi} \cos\theta + \sin\theta\,d\theta$$
$$= \frac{8}{3}\left[\sin\theta - \cos\theta\right]_0^{2\pi}$$
$$= \frac{8}{3}\left[(0-1)-(0-1)\right] = 0$$

(c) The graph of $f(x,y) = x+y$ is a plane through the origin w/ normal vector $(1,1,-1)$. The integral measures signed volume of the solid bounded by the xy-plane, the plane $x+y-z=0$, and the cylinder $x^2+y^2=4$.

See Geogebra

By symmetry, the signed volume is 0.

Activity 11.5.5

- Complete together as a class.

Activity 11.5.5. Consider the circle given by $x^2 + (y-1)^2 = 1$ as shown in Figure 11.5.4.

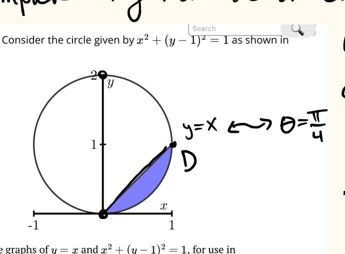


Figure 11.5.4. The graphs of $y = x$ and $x^2 + (y-1)^2 = 1$, for use in Activity 11.5.5.

a. Determine a polar curve in the form $r = f(\theta)$ that traces out the circle $x^2 + (y-1)^2 = 1$. (Hint: Recall that a circle centered at the origin of radius r can be described by the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$.)

b. ~~Find the exact average value of $f(x,y) = \sqrt{x^2+y^2}$ over the interior of the circle $x^2 + (y-1)^2 = 1$.~~ Skip (b)

c. Find the volume under the surface $h(x,y) = x$ over the region D, where D is the region bounded above by the line $y = x$ and below by the circle (this is the shaded region in Figure 11.5.4).

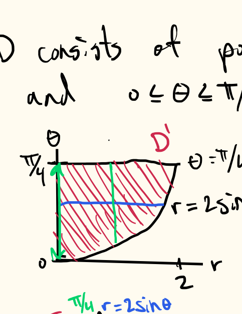
d. Explain why in both (b) and (c) it is advantageous to use polar coordinates.

(a) $x^2 + (y-1)^2 = 1$

Substitute $x = r\cos\theta$, $y = r\sin\theta$

$1 = (r\cos\theta)^2 + (r\sin\theta - 1)^2$
$= r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1$
$= r^2 - 2r\sin\theta + 1$
$\Rightarrow 0 = r(r - 2\sin\theta)$
$\Rightarrow \boxed{r = 2\sin\theta}$

The region D consists of points (r,θ) where $0 \le r \le 2\sin\theta$ and $0 \le \theta \le \pi/4$

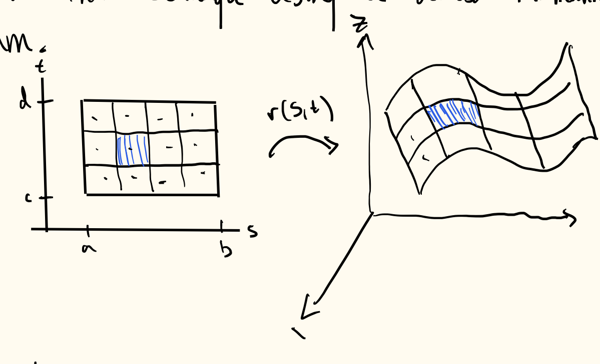


(c)
$$\iint_D x\,dA \overset{\text{Type II}}{\underset{D'}{=}} \int_0^{\pi/4}\int_{r=0}^{r=2\sin\theta} r^2\cos\theta\,dr\,d\theta$$
$$= \frac{1}{3}\int_0^{\pi/4}\left[r^3\cos\theta\right]_0^{2\sin\theta} d\theta$$
$$= \frac{8}{3}\int_0^{\pi/4} \sin^3\theta\cos\theta\,d\theta \qquad u = \sin\theta,\ du = \cos\theta\,d\theta$$
$$= \frac{8}{3}\int_0^{\sqrt{2}/2} u^3\,du = \frac{8}{3}\left[\frac{1}{4}u^4\right]_0^{\sqrt{2}/2}$$
$$= \frac{8}{3}\cdot\frac{1}{4}\cdot\left(\frac{\sqrt{2}}{2}\right)^4$$
$$= 1/6$$

Section 11.6

Section 11.6.2 Surface Area

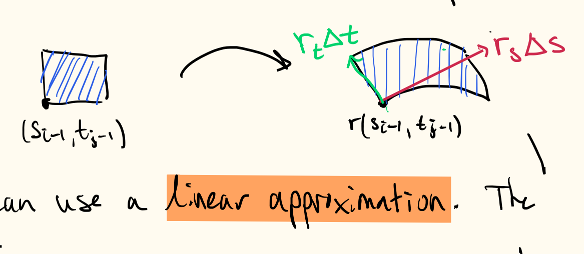
Let $r(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ be a surface over a rectangle $a \le s \le b$, $c \le t \le d$. We can approximate the area of the surface over the rectangle using a double riemann sum.



Step 1.

- Partition $[a,b]$ into m subintervals of equal length $\Delta s = \frac{b-a}{m}$ w/ endpoints $a = s_0 < s_1 < \dots < s_m = b$
- Partition $[c,d]$ into n subintervals of equal length $\Delta t = \frac{d-c}{n}$ w/ endpoints $c = t_0 < t_1 < \dots < t_n = d$.
- this gives us a partition of the rectangle $[a,b]\times[c,d]$ into $m\cdot n$ subrectangles of area $\Delta A = \Delta s \cdot \Delta t$

Step 2 Approximate the area of the wavy parallegrams on the surface



We can use a linear approximation. The vector

$$r_s(s_{i-1},t_{j-1}) = \langle x_s(s_{i-1},t_{j-1}), y_s(s_{i-1},t_{j-1}), z_s(s_{i-1},t_{j-1})\rangle$$

If we move a small distance Δs in the rectangle, we move approximately $r_s\Delta s$ along the surface.

Similarly, a small distance Δt corresponds to approximately $r_t\Delta t$ along the surface.

The vectors $r_s(s_{i-1},t_{j-1})\Delta s$ and $r_t(s_{i-1},t_{j-1})\Delta t$ span a parallegram whose area is approximately the same as the distorted blue parallelogram. The area is

$$|r_s(s_{i-1},t_{j-1})\Delta s \times r_t(s_{i-1},t_{j-1})\Delta t| = |r_s(s_{i-1},t_{j-1}) \times r_t(s_{i-1},t_{j-1})|\Delta s\Delta t$$

To approximate the surface area, we sum them up

$$\sum_{j=1}^{n}\sum_{i=1}^{m} |r_s(s_{i-1},t_{j-1}) \times r_t(s_{i-1},t_{j-1})|\Delta s\Delta t$$

We get a better approximation by increasing the number of subrectangles.

The exact surface area is

$$\lim_{n,m\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{m} |r_s(s_{i-1},t_{j-1}) \times r_t(s_{i-1},t_{j-1})|\Delta s\Delta t.$$

We recognize this limit as a double integral of the function $|r_s \times r_t|$. Thus,

$$\text{Surface Area} = \int_c^d\int_a^b |r_s \times r_t|\,ds\,dt$$

Surface area over a closed and bounded region can also be computed by integrating the same function.

Example Surface Area of a graph.

Suppose $f(x,y)$ is a continuously differentiable function. The graph $z = f(x,y)$ is a surface. You can parametrize it as

$$r(s,t) = \langle s, t, f(s,t)\rangle.$$

Let R be a closed and bounded region.

The surface area of the graph is

$$\iint_R |r_s \times r_t|\,ds\,dt = \iint_R \sqrt{1 + f_x^2 + f_y^2}\,dA$$

$r_s = \langle 1, 0, f_s\rangle$
$r_t = \langle 0, 1, f_t\rangle$

$$|r_s \times r_t| = \left|\begin{vmatrix} i & j & k \\ 1 & 0 & f_s \\ 0 & 1 & f_t \end{vmatrix}\right|$$
$$= \left|\begin{vmatrix} 0 & f_s \\ 1 & f_t \end{vmatrix} i - \begin{vmatrix} 1 & f_s \\ 0 & f_t \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} k\right|$$
$$= |\langle -f_s, -f_t, 1\rangle|$$
$$= \sqrt{f_s^2 + f_t^2 + 1}$$

Activity 11.6.3

- Complete and discuss w/ the person next to you.
- Class discussion.

Activity 11.6.3. Consider the cylinder with radius a and height h defined parametrically by

$$\mathbf{r}(s,t) = a\cos(s)\mathbf{i} + a\sin(s)\mathbf{j} + t\mathbf{k}$$

for $0 \le s \le 2\pi$ and $0 \le t \le h$, as shown in Figure 11.6.7.

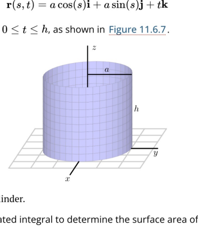


Figure 11.6.7. A cylinder.

a. Set up an iterated integral to determine the surface area of this cylinder.

b. Evaluate the iterated integral.

c. Recall that one way to think about the surface area of a cylinder is to cut the cylinder horizontally and find the perimeter of the resulting cross sectional circle, then multiply by the height. Calculate the surface area of the given cylinder using this alternate approach, and compare your work in (b).

(a) $r_s = \langle -a\sin(s), a\cos(s), 0\rangle$
$r_t = \langle 0, 0, 1\rangle$

$$|r_s \times r_t| = \left|\begin{vmatrix} i & j & k \\ -a\sin(s) & a\cos(s) & 0 \\ 0 & 0 & 1 \end{vmatrix}\right|$$
$$= \left|\begin{vmatrix} i & j \\ -a\sin(s) & a\cos(s) \end{vmatrix}\right|$$
$$= |\langle a\cos(s), a\sin(s), 0\rangle|$$
$$= \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = \sqrt{a^2} = a$$

$$\iint |r_s \times r_t|\,ds\,dt = \int_0^h\int_0^{2\pi} a\,ds\,dt \overset{(b)}{=} 2\pi a h$$

Section 11.7.1 Triple Integrals

Triple integrals can be defined using Riemann sums or via iterated integrals. We will use the latter.

Triple integral of a function $f(x,y,z)$ over a rectangular prism $R = [a,b]\times[c,d]\times[e,f]$

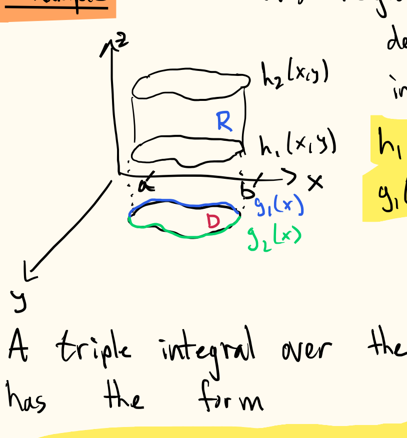
$$\iiint_B f(x,y,z)\,dV = \int_e^f\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

Example $f(x,y,z) = x - y + 2z$ $\quad B = [-2,3]\times[1,4]\times[0,2]$

$$\iiint_B f\,dV = \int_{-2}^{3}\int_1^4\int_0^2 x-y+2z\,dz\,dy\,dx$$
$$= \int_{-2}^{3}\int_1^4 \left[xz - yz + z^2\right]_0^2 dy\,dx$$
$$= \int_{-2}^{3}\int_1^4 2x - 2y + 4\,dy\,dx$$
$$= \int_{-2}^{3}\left[2xy - y^2 + 4y\right]_1^4 dx$$
$$= \int_{-2}^{3} 8x - 16 + 16 - 2x + 1 - 4\,dx$$
$$= \int_{-2}^{3} 6x - 3\,dx = 3x^2 - 3x\Big|_{-2}^{3}$$
$$= 27 - 9 - 12 - 6$$
$$= 0$$

You can evaluate a triple integral over a more general region R as long you can describe it as a solid bounded by the graphs of two continuous functions of two variables over a domain D bounded by two, functions of a single variable.

Example The solid region R below is defined by the inequalities



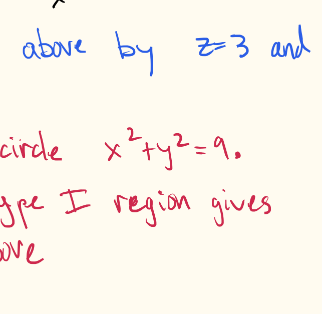
$h_1(x,y) \le z \le h_2(x,y)$
$g_1(x) \le y \le g_2(x)$
$a \le x \le b$

A triple integral over the region R above has the form

$$\iiint_R f(x,y,z)\,dV = \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Example Set up a triple integral over the solid bounded by $z = \sqrt{x^2+y^2}$ and $z = 3$.

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{x^2+y^2}}^{3} f(x,y,z)\,dz\,dy\,dx$$



The region is bounded above by $z=3$ and below by $z = \sqrt{x^2+y^2}$

The region R is the circle $x^2+y^2=9$. Viewing R as a Type I region gives the triple integral above